# 3-Cocycles and the Operator Product Expansion

Javier P. Muniain † and José Wudka ‡

*Department of Physics*
*University of California, Riverside*
*California 92521-0413, U. S. A.*

December 19, 1995

**Abstract**

Anomalous contributions to the Jacobi identity of chromo-electric fields and non-Abelian vector currents are calculated using a non-perturbative approach that combines operator product expansion and a generalization of Bjorken-Johnson-Low limit. The failure of the Jacobi identity and the associated 3-cocycles are discussed.

†muniain@phyun0.ucr.edu ‡jose.wudka@ucr.edu

## 1. Introduction

The study and evaluation of commutators, as well as their algebraic properties has been the motive of much research over the past years. Many results, leading to important measurable effects were found using canonical commutation relations, which, unfortunately, are often ill defined. This was made clear by Schwinger [1] in his evaluation of the matrix element $\langle 0|[J_0(x), J_i(y)]|0\rangle$ ($J^\mu$ denotes a current) at equal time. This commutator has a non-canonical term proportional to the gradient of a delta function which is mandated by locality, Lorentz covariance, positivity and current conservation, and which is not generated following (naive) canonical manipulations.

Much work has been done towards finding perturbative expressions for the commutators [2]. Recently, an effort was made to find a practical method to evaluate commutators in a non-perturbative way [3] based on the operator product expansion (OPE) [4] and on the Bjorken, Johnson and Low (BJL) [5] definition of the commutator. The BJL definition preserves all desirable features of the theory, and reproduces the canonical results whenever these are well defined [6]. In the present paper we generalize the method proposed in [3] to the case of double commutators [1], in particular we will study violations of the Jacobi identity. The present approach is based on a double high-energy limit (taken in a particular order) of the Green function for three local operators.

Given any three operators $A$, $B$ and $C$ we define the quantity

$$\mathcal{J}[A, B, C] = [[A, B], C] + [[B, C], A] + [[C, A], B]. \tag{1}$$

which vanishes whenever the Jacobi identity is preserved. Before we proceed it is worth pointing out that in a theory where all the linear operators are well defined no violations of the Jacobi identity can appear, and $\mathcal{J}$ is identically zero. In this paper we will consider models in which the operators and their products require regularization, for such theories we construct an operator which is naively equal to $\mathcal{J}$ (that is, it coincides with the expression (1) whenever the operator products are well defined), but which has finite matrix elements and respects all the desirable symmetries of the model. The price is that not all such matrix elements need vanish. The procedure we describe below provides a *definition* of $\mathcal{J}$.

Situations in which $\mathcal{J} \neq 0$ present problems in providing well-defined representations for the corresponding algebra of operators. A non vanishing $\mathcal{J}$ is then understood as an obstruction in constructing such representations in terms of operator-valued distributions [11]. However, objects which are local in time and obey $\mathcal{J} \neq 0$ may still be defined in terms of their commutators with space-time smoothed operators.

The expression we obtain for $\mathcal{J}$ depends on a small number of undetermined constants. The present method is not powerful enough to determine whether such

[1] For a related publication see Ref. [7].

constants are non-zero. Nonetheless it is still possible to obtain some non-trivial information concerning the expression for our definition of $\mathcal{J}$ mainly based on the consistent implementation of the model's symmetries. We will comment on this fact in the last section.

It is well known [8, 9, 10] that violations of the Jacobi identity $\mathcal{J} = 0$ within an algebra generate, in general, violations of associativity in the corresponding group. If the group generators, denoted by $G_a$, satisfy

$$\mathcal{J}\left[G_{a_1}, G_{a_2}, G_{a_3}\right] = \frac{i}{3!}\omega_{[a_1a_2a_3]} \neq 0 \tag{2}$$

($[a_1a_2a_3]$ denotes antisymmetrization in all variables $a_i$) the corresponding lack of associativity is parameterized by the three-cocycle $\omega$ (for a review see Refs. [8]). Consistency requires the closure relation [9]

$$f_{c[a_1a_2}\omega_{a_3a_4]c} = 0 \tag{3}$$

(where summation over $c$ is understood).

The existence and properties of 3-cocycles has been under investigation in quantum field theory for some time now. The behavior of gauge transformations in an anomalous gauge theory, as well as in a consistent gauge theory with Chern-Simons term, can be given a unified description in terms of cocycles [12]. Violations of the Jacobi identity also appear in the quark model: if the Schwinger term in the commutator between time and space components of a current is a c-number, the Jacobi identity for triple commutators of spatial current components must fail [13]. This fact has been verified in perturbative BJL calculations [14]

In the context of quantum mechanics, 3-cocycles appear in the presence of magnetic monopoles [10]. For example, a single particle moving in a magnetic field $\vec{B}$ satisfies $J[v^1, v^2, v^3] = (e\hbar^2/m^3)\vec{\nabla}\cdot\vec{B}$, where $v^i$ represent the components of the (gauge invariant) velocity operator. If $\vec{\nabla}\cdot\vec{B} \neq 0$, as in the case of a point monopole, the Jacobi identity fails.

The paper is organized in the following manner. Section 2 is dedicated to the description of the method. Section 3, as an application of the method, studies the failure in field theory of the Jacobi identity for chromo-electric fields. Following this, Section 4 is dedicated to 3-cocycles associated to the QCD quark charges and Gauss' law generators. The results of these sections are compared to the results derived form perturbation theory in section 5. Conclusions are presented in Section 6.

## 2. Description of the method

In this section we will generalize the method proposed in [3] to study double commutators and the possibility of violation of Jacobi identity. The canonical evaluation

of equal time commutators sometimes presents ambiguities [1], and it becomes necessary to have an alternative way to define and calculate these objects. This is achieved by the Bjorken, Johnson and Low [5] *definition* of the single commutators (for a review see [6]) which relies only on the construction of the time-ordered product of the operators whose commutator is desired. Specifically, the commutator of $A$ and $B$ is obtained from

$$\lim_{p^0\to\infty} p^0 \int d^n x\; e^{ipx} \langle\alpha|TA(x/2)B(-x/2)|\beta\rangle = \\ = i\int d^{n-1}x\; e^{-i\vec{p}\cdot\vec{x}}\langle\alpha|[A(0,\vec{x}/2),B(0,-\vec{x}/2)]|\beta\rangle. \tag{4}$$

where $p^0$ stands for the time component of the four-momentum. The BJL definition (4) uses the time ordered product $T$, which (in general) is not a Lorentz covariant object [6], while in field theory (e.g. Feynman diagrams in perturbation theory) one calculates an associated covariant object, usually denoted by $T^*$. The difference between $T$ and $T^*$ is local in time, involving $\delta(x_0)$ and its derivatives [6], which translates into a polynomial in $p^0$ in momentum space. Therefore in (4) we can replace $T$ by $T^*$ provided we drop all polynomials in $p^0$. Equivalently, the Fourier transform of the commutator is the residue of the $1/p^0$ term in a Laurent expansion of the time ordered product $T^*$ (divided by $i$).

This approach can easily be extended to the study of double commutators. We first define

$$\mathcal{C}(p,q) = \int d^nxd^ny\, e^{i(px+qy)}\langle\alpha|TA(x)B(y)C(0)|\beta\rangle, \tag{5}$$

and use the (formal) identities

$$\begin{aligned} \frac{\partial}{\partial x_0}TA(x)B(y)C(0) &= T(\dot{A}BC)+\delta(x_0-y_0)T([A,B]_{(x_0)}C(0)) \\ &+ \delta(x_0)T(B(y_0)[A,C]_{(x_0)}) \end{aligned} \tag{6}$$

$$\begin{aligned} \frac{\partial}{\partial y_0}TA(x)B(y)C(0) &= T(A\dot{B}C)+\delta(y_0-x_0)T([B,A]_{(y_0)}C(0)) \\ &+ \delta(y_0)T(A(x_0)[B,C]_{(y_0)}), \end{aligned} \tag{7}$$

where the subscript in the commutators indicates the common time of the operators.

To simplify the resulting expressions we define

$${}_q\mathbb{L}_p = \lim_{q_0\to\infty} q_0 \quad \lim_{p_0\to\infty;q_0=\text{const}} p_0 \tag{8}$$

and obtain, after straight-forward manipulations,

$$\begin{aligned} {}_q\mathbb{L}_p\mathcal{C}(p,q) &= \int d^{n-1}x\; d^{n-1}y\; e^{-i(\vec{p}\cdot\vec{x}+\vec{q}\cdot\vec{y})}\langle\alpha|[B(0,\vec{y}),[C(0),A(0,\vec{x})]]|\beta\rangle \\ {}_p\mathbb{L}_k\mathcal{C}(p,-p-k) &= \int d^{n-1}x\; d^{n-1}y\; e^{-i(\vec{p}\cdot\vec{x}+\vec{q}\cdot\vec{y})}\langle\alpha|[A(0,\vec{x}),[B(0,\vec{y}),C(0)]]|\beta\rangle \\ {}_k\mathbb{L}_q\mathcal{C}(-q-k,q) &= \int d^{n-1}x\; d^{n-1}y\; e^{-i(\vec{p}\cdot\vec{x}+\vec{q}\cdot\vec{y})}\langle\alpha|[C(0),[A(0,\vec{x}),B(0,\vec{y})]]|\beta\rangle, \end{aligned} \tag{9}$$

where $k = -p - q$. These expressions imply

$$({}_q\mathbb{L}_p + {}_p\mathbb{L}_k + {}_k\mathbb{L}_q)\,\mathcal{C} = \int d^{n-1}x d^{n-1}y e^{-i\vec{p}\cdot\vec{x} - i\vec{q}\cdot\vec{y}} \mathcal{J}[A, B, C] \tag{10}$$

where, as above, $\mathcal{J}[A, B, C] = [A, [B, C]] + [B, [C, A]] + [C, [A, B]]$.

The above manipulations suggest that we *define* $\mathcal{J}[A, B, C]$ via (10) [2]. In the following we will use this definition of $\mathcal{J}$

Since we are interested in the large-momentum-transfer behavior, it is appropriate to express the product of operators in (5) as a sum of non-singular local operators with possibly singular c-number coefficients [4],

$$\int d^n x d^n y \; e^{i(px+qy)} \langle \alpha | T A(x) B(y) C(0) | \beta \rangle = \sum_i c_i(p, q) \; \langle \alpha | \mathcal{O}_i(0) | \beta \rangle, \tag{11}$$

each term in the OPE should respect the same symmetries (and possess the same internal quantum numbers) as the Green's function (5). As for the single commutator case [3] it is more convenient to derive the various (double) commutators from the covariant time-ordered product $T^*$. the difference $T[A(x)B(y)C(z)] - T^*[A(x)B(y)C(z)]$ is an operator local in $x-y$ or $y-z$ or $x-z$. Thus we will drop all terms proportional to a polynomial in $p^0$, $q^0$ or $k^0$.

Substituting (11) in (10) we obtain

$$\begin{aligned}\int d^{n-1}x \; d^{n-1}y \; e^{-i(\vec{p}\cdot\vec{x} + \vec{q}\cdot\vec{y})} \langle \alpha | \mathcal{J}[A(0, \vec{x}), B(0, \vec{y}), C(0)] | \beta \rangle = \\ = \sum_i ({}_q\mathbb{L}_p + {}_p\mathbb{L}_k + {}_k\mathbb{L}_q)\, c_i(p, q) \langle \alpha | \mathcal{O}_i(0) | \beta \rangle\end{aligned} \tag{12}$$

It is worth pointing out that similar manipulations have been used to provide constraints on the general form of current anomalies [15].

## 3. Jacobi Identity for chromo-electric fields

In this section we investigate the existence of 3-cocycles associated with the (chromo) electric fields of a gauge theory, denoted by $E_i^a = F_{0i}^a$, where $F_{\mu\nu}^a$ is the non-Abelian gauge field strength. We will consider the four-dimensional case first and then briefly consider the case of two dimensions.

We evaluate the Jacobi operator for three chromo-electric fields by studying the behavior of the correlator of three strength tensors $T\left\{ F_{\mu_1\nu_1}^{a_1}(x_1) F_{\mu_2\nu_2}^{a_2}(x_2) F_{\mu_3\nu_3}^{a_3}(x_3) \right\}$ for the case of $\mu_r = 0$, $\nu_r \neq 0$.

Following (5) we consider

$$\mathcal{C}^{a_1 a_2 a_3}_{\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3}(k_1, k_2) = \int d^4x_1 d^4x_2 \, e^{i(k_1x_1 + k_2x_2)} \left\langle \alpha \left| T^* \left\{ F_{\mu_1\nu_1}^{a_1}(x_1) F_{\mu_2\nu_2}^{a_2}(x_2) F_{\mu_3\nu_3}^{a_3}(0) \right\} \right| \beta \right\rangle, \tag{13}$$

[2]When canonical manipulations are well defined we will have $\mathcal{J}[A, B, C] = 0$.

which must be symmetric under $(k_r, \mu_r, \nu_r, a_r) \leftrightarrow (k_s, \mu_s, \nu_s, a_s)$, and antisymmetric under $(\mu_r, \nu_r) \leftrightarrow (\nu_r, \mu_r)$ where $r, s = 1, 2, 3$. In order to present the expressions symmetrically we define

$$k_3 = -k_1 - k_2. \tag{14}$$

The canonical mass dimension of $\mathcal{C}$ equals $-2$ which implies that the only terms in the OPE which survive the double limits are proportional to the identity operator [3],

$$\mathcal{C}^{a_1 a_2 a_3}_{\mu_1 \nu_1 \mu_2 \nu_2 \mu_3 \nu_3} = c^{a_1 a_2 a_3}_{\mu_1 \nu_1 \mu_2 \nu_2 \mu_3 \nu_3} \, \mathbf{1} + \cdots , \tag{15}$$

where the remaining terms will not contribute to the final result. The Wilson coefficients multiplying the identity operator will be such that $[c^{a_1 \cdots}_{\mu_1 \cdots}] = (\text{mass})^{-2}$.

The coefficient function $c$ consists of a sum of terms each of which takes the form

$$\frac{\overbrace{k \otimes \cdots \otimes k}^{n \text{ factors}}}{(\text{polynomial of degree } l \text{ in the } k_i^2)} \tag{16}$$

For the present calculation we must have $n = 2(l-1)$.

In restricting the values of $n$ note first that all terms of the form $k_i \cdot k_j$ can be turned into a linear combination of the $k_i^2$ by using $k_1 + k_2 + k_3 = 0$; also note that multiplying the above expression by a dimensionless function will, at most, modify the final result by an overall multiplicative constant, thus we can replace (for $l > m$)

$$\frac{(\text{polynomial of degree } m \text{ in } k_i^2)}{(\text{polynomial of degree } l \text{ in } k_i^2)} \rightarrow \frac{1}{(\text{polynomial of degree } l-m \text{ in } k_i^2)} \tag{17}$$

which implies that we can ignore all contributions to $c$ containing factors of the form $k_i \cdot k_j$ in the numerator. Using this and the fact that there are six "external" indices $\mu_{1,2,3}$, $\nu_{1,2,3}$ and noting that we need include at most one $\epsilon$ tensor, we find that we can restrict ourselves to $n = 0, 2, 4, 6$. We will consider the case $n = 0$ in detail, the others can be treated in the same way.

The coefficient corresponding to $n = 0$ in (16) takes the form

$$\sum_\pi \tau_{\mu_{\pi 1} \nu_{\pi 1} \mu_{\pi 2} \nu_{\pi 2} \mu_{\pi 3} \nu_{\pi 3}} u^{a_{\pi 1} a_{\pi 2} a_{\pi 3}} \left( \sum_r x_r k_{\pi r}^2 \right)^{-1} \tag{18}$$

where $\pi$ denotes a permutation of $1, 2, 3$; the summation is over the $3!$ such permutations. The tensor $\tau$ is constructed out of the metric and the $\epsilon$ tensor. Since the tensor $u$ takes values on a Lie algebra, its general expression will be of the form

$$u^{abc} = u_1 f^{abc} + u_2 d^{abc}, \tag{19}$$

[3] As in [3], we assume that the sum of three double commutators is a renormalization group invariant quantity.

where $f$ denotes the (completely antisymmetric) group structure constants and $d^{abc}$ denotes the completely symmetric object $\mathrm{tr} T^a\{T^b, T^c\}$ ($T^a$ denote the group generators).

Consider now the limit ${}_{k_r}\mathbb{L}_{k_s}$, abbreviated ${}_r\mathbb{L}_s$, and let $u$ be the (unique) index $\neq r,\, s$. The polynomial in the denominator can be written

$$(\tilde{x}_r + \tilde{x}_u)\, k_r^2 + (\tilde{x}_s + \tilde{x}_u)\, k_s^2 + 2\tilde{x}_u\, k_r \cdot k_s; \qquad (u \neq r, s) \tag{20}$$

where $\tilde{x}_r = x_{\pi^{-1}r}$ and where we used $\sum_r x_r k_{\pi r}^2 = \sum_r x_{\pi^{-1}r} k_r^2$. Then we have

$${}_r\mathbb{L}_s \frac{1}{\sum_r \tilde{x}_r k_r^2} = \frac{1}{2\tilde{x}_u}\, \delta_{\tilde{x}_s + \tilde{x}_u} \tag{21}$$

where $\delta_{x+x'}$ denotes the Kronecker delta. The above expression implies

$$\left({}_1\mathbb{L}_2 + {}_2\mathbb{L}_3 + {}_3\mathbb{L}_1\right) \frac{1}{\sum_r \tilde{x}_r k_r^2} = \frac{1}{2}\left[\frac{\delta_{\tilde{x}_2+\tilde{x}_3}}{\tilde{x}_3} + \frac{\delta_{\tilde{x}_3+\tilde{x}_1}}{\tilde{x}_1} + \frac{\delta_{\tilde{x}_1+\tilde{x}_2}}{\tilde{x}_2}\right] = A(\tilde{x}_1, \tilde{x}_2, \tilde{x}_3) \tag{22}$$

$A$ is a completely antisymmetric function of the $\tilde{x}$ and so

$$A(\tilde{x}_1, \tilde{x}_2, \tilde{x}_3) = \nu_\pi\; A\,(x_1, x_2, x_3) \tag{23}$$

where $\nu_\pi$ denotes the signature of the permutation $\pi$.

Note that $A$ vanishes unless the sum of two of the parameters $\tilde{x}$ is zero which is not usually realized, within perturbation theory. This can be understood by noting that expression (18) will present poles whenever $A \neq 0$ and $k_i^0 \propto k_j^0$ (neglecting the spatial components since $k_i^0 \to \infty$). In the vicinity of such poles the Wilson coefficient behaves as $1/(k_i^2 - \eta^2 k_j^2)$ with $\eta$ a real constant depending on the $x_r$. Such behavior is rarely generated within perturbation theory [16]; the high-energy behavior of the triangle graph is, however, an exception [17].

The term under consideration then contributes to the sum of the three double limits the quantity

$$A\,(x_1, x_2, x_3) \sum_\pi \nu_\pi\; \tau_{\mu_{\pi1}\nu_{\pi1}\mu_{\pi2}\nu_{\pi2}\mu_{\pi3}\nu_{\pi3}} u^{a_{\pi1}a_{\pi2}a_{\pi3}} \tag{24}$$

For the case of interest $\mu_r = 0$ and $\nu_r = i_r \neq 0$ whence, of all possible contributions to $\tau$, only the term containing the $\epsilon$ tensor contributes. This leads to a term proportional to the three-dimensional antisymmetric tensor,

$$\tau_{0\, i_{\pi1} 0\, i_{\pi2} 0\, i_{\pi3}} = \tilde{\tau} \epsilon_{i_{\pi1} i_{\pi2} i_{\pi3}} = \tilde{\tau} \nu_\pi\, \epsilon_{i_1 i_2 i_3} \tag{25}$$

for some constant $\tilde{\tau}$. The contribution to the limits then becomes

$$A\, \tilde{\tau}\, \epsilon_{i_1 i_2 i_3} \sum_\pi u^{a_{\pi1}a_{\pi2}a_{\pi3}} = \bar{\tau} \epsilon_{i_1 i_2 i_3} d^{a_1 a_2 a_3} \tag{26}$$

where we used the expression (19) and $\bar{\tau} = 6u_2\tilde{\tau}\, A(x_1, x_2, x_3)$. The terms containing $f^{abc}$ in (19) do not contribute.

The other cases, $n = 1, 2, 3$, although more involved, yield the same type of expressions. Collecting all results we obtain

$$\mathcal{J}\left[E^{a_1}_{i_1}(\vec{x}), E^{a_2}_{i_2}(\vec{y}), E^{a_3}_{i_3}(\vec{z})\right] = \bar{c}\, \epsilon_{i_1 i_2 i_3} \mathrm{Tr}\{T^{a_1}, T^{a_2}\} T^{a_3}\, \delta^3(\vec{x}-\vec{y})\delta^3(\vec{x}-\vec{z}). \qquad (27)$$

where $\bar{c}$ is an undetermined constant. We note that this result also satisfies the closure relation (3). A similar expression was obtained in Ref. [18]; we will compare the present approach with the one followed in this reference in section 4.

For the two-dimensional case only the terms containing $F^b_{\alpha\beta}$ in the OPE contribute to the double limits. The coefficient functions take the same form as in (16) where now we have $n \leq 4$. After a short calculation we obtain

$$\mathcal{J}\left[E^{a_1}(\vec{x}), E^{a_2}(\vec{y}), E^{a_3}(\vec{z})\right] = \bar{c}'\, u^{a_1 a_2 a_3 b} E^b\, \delta^3(\vec{x}-\vec{y})\delta^3(\vec{x}-\vec{z}); \qquad (1+1 \text{ dim.}) \quad (28)$$

where $u^{abcd}$ is antisymmetric in its first three indices and must be constructed out of traces of products of generators,

$$u^{a_1 a_2 a_3 b} = i\left[f_{a_1 a_2 c} d_{c a_3 b} + f_{a_2 a_3 c} d_{c a_1 b} + f_{a_3 a_1 c} d_{c a_2 b}\right] \qquad (29)$$

Using this expression (28) is seen to satisfy (3).

## 4. The 3-cocycle in current algebra

We now follow the above procedure to study the Jacobi identity for three non-Abelian charges. We start from a gauge theory with anti-hermitian generators $\{T^a\}$ and assume that a set of current operators $J^a_\mu$ can be defined (we will not need to specify the chirality properties of these currents). We then consider the operator

$$\mathcal{C}^{abc}_{\mu\nu\rho}(k_1, k_2) = \int d^4x d^4y e^{i(k_1 x + k_2 y)} T^* \left\{ J^{a_1}_{\mu_1}(x) J^{a_2}_{\mu_2}(y) J^{a_3}_{\mu_3}(0) \right\}, \qquad (30)$$

which is symmetric under any permutation $(k_s,\, \mu_s,\, a_s) \to (k_r,\, \mu_r,\, a_r)$ for $r, s = 1, 2, 3$. As in the previous section we consider first the four-dimensional case and then briefly state the results for the two-dimensional theory.

We now expand $\mathcal{C}^{abc}_{\mu\nu\rho}$ in a series of local operators. The terms that will contribute to the double limits are proportional to the operators $\mathbf{1}$, $F^a_{\mu\nu}$, $\tilde{F}^a_{\mu\nu}$, $J^b_\alpha$, $(D_\alpha F_{\beta\gamma})^b$, and $\left(D_\alpha \tilde{F}_{\beta\gamma}\right)^b$. The general expressions for arbitrary values of the indices are quite involved and not very illuminating; we will therefore consider only two cases: the terms proportional to the unit operator (corresponding to the vacuum expectation value of $\mathcal{J}$), and the case $\mu_i = 0$ which can lead to violations of the Jacobi identity in the global algebra generated by the charges.

### 4.1 Terms proportional to 1

We consider the Wilson coefficient associated with the unit operator first. Using the same arguments as for the previous section we conclude that the Wilson coefficient should take the same form as in (16) with $l = 1, n = 3$, explicitly

$$c_{\mathbf{1}} = \sum_{r,s,t,\pi} \tau^{rst}_{\mu_{\pi 1}\mu_{\pi 2}\mu_{\pi 3}\alpha\beta\gamma} \, k^{\alpha}_{\pi r} k^{\beta}_{\pi s} k^{\gamma}_{\pi t} \, u^{a_{\pi 1} a_{\pi 2} a_{\pi 3}}_{rst} \left( \sum_u x^{rst}_u k_{\pi u} \right)^{-1} \tag{31}$$

The evaluation of the three double limits is essentially the same as for the previous case and we will omit the details. We obtain

$$\begin{aligned} \left({}_1\mathbb{L}_2 + {}_2\mathbb{L}_3 + {}_3\mathbb{L}_1\right) c_{\mathbf{1}} \;\; = \;\; & \sum \nu_\pi \, \bar\tau^{d;rst}_{\mu_{\pi 1}\mu_{\pi 2}\mu_{\pi 3} ijn} \, k^i_{\pi r} k^j_{\pi s} k^n_{\pi t} \, d_{a_1 a_2 a_3} \\ & + \sum \bar\tau^{f;rst}_{\mu_{\pi 1}\mu_{\pi 2}\mu_{\pi 3} ijn} \, k^i_{\pi r} k^j_{\pi s} k^n_{\pi t} \, f_{a_1 a_2 a_3} \end{aligned} \tag{32}$$

where $\nu_\pi$ is the signature of the permutation $\pi$.

The tensors $\bar\tau^d$ and $\bar\tau^f$ must be constructed out of the metric and the $\epsilon$ tensor. This implies that the result vanishes when $\mu_i = 0$, *i.e.* $\left\langle 0 \left| \mathcal{J}\left[ J^a_0 J^b_0 J^c_0 \right] \right| 0 \right\rangle = 0$, as verified by explicit perturbative calculations [14]. If we consider the case $\mu_i = j_i \neq 0$ we obtain terms $\sim k^{j_1}_r k^{j_2}_s k^{j_3}_t$ and $\sim k^{j_1}_r \delta_{j_1 j_2} \vec{k}^2_s$ proportional to $f_{a_1 a_2 a_3}$ which also agree with the results obtained perturbation theory [14].

### 4.2 Jacobi identity for the global current algebra

In studying violations of the Jacobi identity in the algebra of the non-Abelian charges we define

$$Q^a = \int d^3\vec{x} \, J^a_0 \tag{33}$$

so that when calculating $\mathcal{J}\left[Q^{a_1}, Q^{a_2}, Q^{a_3}\right]$ (see (1)) we need only consider only the case $\mu_{1,2,3} = 0$.

In the previous subsection we showed that there are no contributions to the operator $\mathcal{J}[J^{a_1}_0 J^{a_2}_0 J^{a_3}_0]$ proportional to the unit operator. The contributions proportional to the operators $F$ and $\tilde F$ have Wilson coefficients of the same form as in (16) with $n = 2l - 1, \; l = 1, 2, 3$. When $\mu_i = 0$ the various terms resulting form the three double limits are proportional to $\vec{k}_r \cdot \vec{E}^b$ or $\vec{k}_r \cdot B^b$, $(r = 1, 2, 3)$. Thus they will not contribute to the global algebra (for which we set $\vec{k}_r = 0$).

Next we consider the Wilson coefficients associated with $J^b_\alpha$. These again take the form (16) with $n = 2(l - 1), \; l = 1, 2, 3$. As an example we study the $l = 1$ case; the explicit form of the coefficient function is

$$c^{a_1 a_2 a_3 b}_{J;\mu_1\mu_2\mu_3\alpha} = \sum_\pi \tau_{\mu_{\pi 1}\mu_{\pi 2}\mu_{\pi 3}\alpha} u^{a_{\pi 1} a_{\pi 2} a_{\pi 3} b} \left( \sum_r x_r k^2_{\pi r} \right)^{-2} \tag{34}$$

where, as above, $\pi$ denotes a permutation of 1,2,3 and $u^{abcd}$ is constructed from the traces of four group generators with all possible orderings. Evaluation of the contribution to the three double limits is almost identical to the one described above. As a result we get

$$({}_1\mathbb{L}_2 + {}_2\mathbb{L}_3 + {}_3\mathbb{L}_1)\, c_J = A(x_1, x_2, x_3) \sum_\pi \nu_\pi \tau_{\mu_{\pi 1}\mu_{\pi 2}\mu_{\pi 3}\alpha} u^{a_{\pi 1} a_{\pi 2} a_{\pi 3} b} \tag{35}$$

with $A$ defined in (22).

Using then $\tau_{000\alpha} = \bar{\tau} g_{\alpha 0}$ for some constant $\bar{\tau}$, we find that the term containing $J^a_\mu$ in (30) contributes the operator

$$\left( \bar{\tau}\, A \sum_\pi \nu_\pi u^{a_{\pi 1} a_{\pi 2} a_{\pi 3} b} \right) Q^b. \tag{36}$$

to $\mathcal{J}[Q_{a_1} Q_{a_2} Q_{a_3}]$. The $n = 2$ and $n = 4$ cases yield expressions of the same form.

Finally we consider the contributions proportional to the operators $D_\alpha F_{\beta\gamma}$. In this case the coefficients are of the form (16) with $n = 2(l-1)$, $l \le 4$. Again concentrating on the charge operators we require $\mu_{1,2,3} = 0$ and obtain, following the procedure outlined in the previous section, that the contribution to the sum of the three double limits is of the form

$$\text{const} \int d^3\vec{x}\, (D^\mu F_{\mu 0})^b \sum_\pi \nu_\pi u^{a_{\pi 1} a_{\pi 2} a_{\pi 3} b}. \tag{37}$$

An identical procedure can be followed for $D\tilde{F}$. The resulting expressions contain $D^\mu \tilde{F}_{\mu 0}$ and vanish by virtue of the Bianchi identities.

Collecting the above results we conclude that

$$\mathcal{J}\left[Q^{a_1}, Q^{a_2}, Q^{a_3}\right] = \left[ \bar{c}_J Q^b + \bar{c}_{DF} \left( \int d^3\vec{x} D^\mu F_{\mu 0} \right)^b \right] \sum_\pi \nu_\pi u^{a_{\pi 1} a_{\pi 2} a_{\pi 3} b}. \tag{38}$$

for some constants $\bar{c}_J$ and $\bar{c}_{DF}$. Noting that $u$ must be constructed out of traces of generators, and using the Jacobi identity for the generators we obtain that this tensor takes the form (29). It is easy to see that (38) satisfies (3).

If we use the equations of motion $D^\mu F_{\mu\nu} = J_\nu$ and define $\bar{c} = \bar{c}_J + \bar{c}_{DF}$ we obtain

$$\mathcal{J}\left[Q^{a_1}, Q^{a_2}, Q^{a_3}\right] = \bar{c} Q^b \left[ f_{a_1 a_2 c} d_{c a_3 b} + f_{a_2 a_3 c} d_{c a_1 b} + f_{a_3 a_1 c} d_{c a_2 b} \right] \tag{39}$$

For example, for $SU(3)$ we have $\mathcal{J}\left[Q^1, Q^2, Q^3\right] = (\bar{c}\sqrt{3}/2) Q^8$.

We can follow exactly the same procedure for the Gauss' identity operators

$$G^a = Q^a - \int d^3\vec{x}\, (D^\mu F_{\mu\nu})^a \tag{40}$$

assuming that these operators close into an algebra we obtain that the Jacobi identity is violated,

$$\mathcal{J}\left[G^{a_1}, G^{a_2}, G^{a_3}\right] = \bar{c}' G^b \left[f_{a_1 a_2 c} d_{c a_3 b} + f_{a_2 a_3 c} d_{c a_1 b} + f_{a_3 a_1 c} d_{c a_2 b}\right] \tag{41}$$

for those cases where $\bar{c}' \neq 0$. Note however that in the physical subspace, which is annihilated by the $G^a$, the Jacobi identity is valid (this would not be true is the $G^a$ fail to close into an algebra under commutation).

### 4.2 1+1 dimensions

For the two-dimensional case some of the metric tensors which appear in the Wilson coefficients can be replaced by the antisymmetric tensor $\epsilon_{\mu\nu}$. Expressing the results in terms of the left and right-handed currents $J_{L,R} = J_0 \mp J_1$ we obtain

$$\left\langle 0 \left| \mathcal{J}\left[J^a_{h_1} J^b_{h_2} J^c_{h_3}\right] \right| 0 \right\rangle = \sum_r \left[c^r_{d;h_1 h_2 h_3}\, d_{abc} + c^r_{f;h_1 h_2 h_3}\, f_{abc}\right] K_r \tag{42}$$

where $h_i = L, R$ and $K_r$ denotes the spatial component of $k_r$.

Turning now to the global current algebra one can easily verify that the terms in the OPE containing the operators $\tilde{F}^a$ do not generate violations of the Jacobi identity. In contrast, terms containing the operators $J^a_\mu$ do contribute. A straightforward calculation (almost identical to the one described in the case of 4 dimensions) gives

$$\mathcal{J}\left[Q^{a_1}, Q^{a_2}, Q^{a_3}\right] = \left(\sum_{h=L,R} \bar{c}_h Q^b_h\right) \left[f_{a_1 a_2 c} d_{c a_3 b} + f_{a_2 a_3 c} d_{c a_1 b} + f_{a_3 a_1 c} d_{c a_2 b}\right] \tag{43}$$

where $Q^a_h = \int dx\, J^a_h$.

## 5. Perturbative calculations.

The previous results can be compared to the results obtained using perturbation theory. For the vacuum expectation value of the operator $\mathcal{J}[JJJ]$ in 4 dimensions the results are known [14] and agree with the results obtained in sect. 4. The origin of the non-vanishing contribution of such a graph can be traced to the peculiar behavior of the discontinuities of the form factors for the triangle graph [17].

The situation is different when we consider the graphs contributing to $\mathcal{J}[EEE]$ calculated in section 3.

The expression for $\bar{c}$ in (27) can be derived from perturbation theory by obtaining the corresponding Wilson coefficients. To this end we first note that the vertices corresponding to the (composite) operator $F^{\mu\nu}$ are of the form

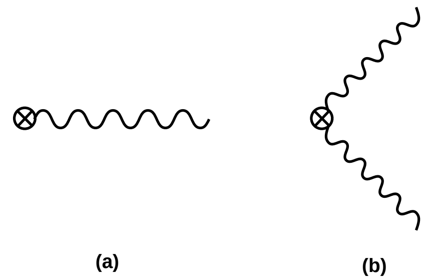


*Fig.* 1 Vertices in the composite operator $F_{\mu\nu}$

The one loop contributions are given by the graph in Fig. 2 which, however, does not contribute to the double limits. In fact, it is a straightforward exercise to show that any graph with one or more of the vertices of type (a) in Fig. 1 with one gauge boson line will not contribute to the three limits. This implies that the leading contributions to $\bar{c}$ are at least $O(g^7)$, where $g$ is the gauge coupling constant, and occur at the three loop level. We will not evaluate the corresponding graphs in this paper.

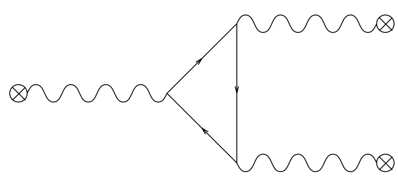

*Fig.* 2 One loop graph contributing to the Jacobi identity for three electric fields.

The absence of perturbative contributions, at least at low orders, to (27) contradicts the results obtained in [18] where (27) was obtained by first calculating the anomalous commutator of two electric fields and then using canonical commutation relations in the evaluation of the Jacobi operator (1). In that calculation the commutator of two electric fields was found to be proportional to the gauge field, which raises questions about the gauge invariance of the result [4]. In view of these problems we revisit calculation of the commutator of two electric fields following the approach described in [3]. We define

$$\mathcal{T}^{\mu\nu\alpha\beta}(p) = \int d^4x\, e^{-ip\cdot x} F^{\mu\nu}(x/2)\, F^{\alpha\beta}(-x/2) \tag{44}$$

[4] A similar situation was discussed in [3]. Note, however, that gauge invariance is not an issue when considering anomalous theories.

whose OPE takes the form $\tau^{\mu\nu\alpha\beta}(p)\mathbf{1} + \cdots$ where $\tau$ is a tensor constructed from $p$, the metric and the $\epsilon$ tensor; the term proportional to the unit operator is the only one that contributes to the BJL limit.

The terms in $\tau$ that generate a non-trivial BJL limit are of the form $g^{\mu\alpha}p^\nu p^\beta/p^2 \pm$ perms, and $\epsilon^{\mu\nu\alpha\gamma}p_\gamma p^\beta/p^2 \pm$ perms; where "perms" denotes similar terms with the indices exchanged to insure antisymmetry under $\mu \leftrightarrow \nu$, and $\alpha \leftrightarrow \beta$; and symmetry under the exchange $(\mu\nu; p) \leftrightarrow (\alpha\beta; -p)$. The commutator of two electric fields is obtained from the limit

$$\lim_{p_0\to\infty} p^0 \tau^{0i0j} \tag{45}$$

which, using the above expression for $\tau$ is seen to vanish. We therefore conclude that

$$\left[E_a^i(\vec{x}/2),\, E_a^j(-\vec{x}/2)\right] = 0 \tag{46}$$

which disagrees with the results of [18].

We believe that this discrepancy is due to the following. In [18] the commutator was computed from the seagull for the current-current commutator by using Ampère's law to relate the current to $\dot{E}_i^a$. The problem with this calculation is that the seagull is not unique, always being defined up to a covariant local contribution [6]. So, if we follow the procedure described in [18] but add to the seagull the covariant contribution

$$\sigma^{\mu\nu}_{\mathrm{cov}\ ab}(x,y) = \frac{i\xi}{24\pi^2}\mathrm{Tr}\{T^a, T^b\}\,\epsilon^{\mu\nu\alpha\beta}A^b_\alpha(y)\partial_\beta\delta^{(4)}(x-y) \tag{47}$$

the final result is the one in [18] multiplied by $1-\xi$. The calculation using the OPE shows that, in fact, $\xi = 1$. This also implies that the expression (27) cannot be derived from (1) by evaluating some commutators canonically and others using the BJL limit.

We now consider the perturbative evaluation of the Jacobi identity for three currents. Following our approach we will be interested in the Wilson coefficient for the term $J^b_\alpha$ in the OPE of the operator $T^*\{JJJ\}$. The one loop contributions to the OPE are obtained from the graphs in Fig. 3.

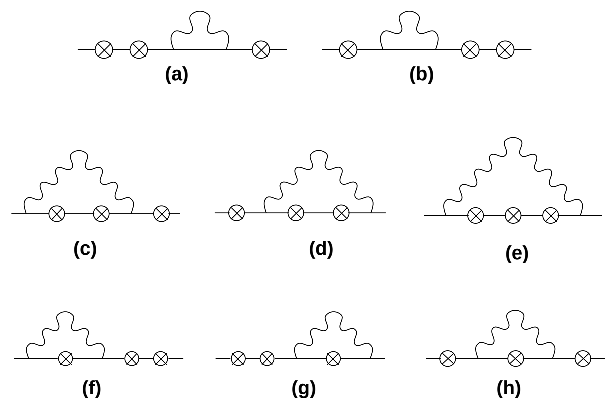


*Fig.* 3 One loop graphs contributing to the Jacobi identity for three currents.

In this calculation of $\mathcal{J}$ the contribution from graphs (cf. fig. 3) (a) and (b) cancel each other; similarly graphs (c) and (d) cancel, while the contributions of (f), (g) and (h) add up to zero. Graph (e) requires careful evaluation; we chose to regulate the theory using a higher covariant derivative method [19] in the gauge-boson sector [5]. The propagator then becomes (in the Feynman gauge) $g_{\mu\nu}/[p^2(1-p^2/\Lambda^2)]$ and we obtain that

$$ {}_j\mathbb{L}_i \,\{\text{graph 3(e)}\} = \frac{g^2}{16\pi^2}\left(\lambda^b\,[\lambda^{a_j}\,,\,[\lambda^{a_k}\,,\,\lambda^{a_i}]]\,\lambda^b\right)\,\ln\frac{\Lambda^2}{m^2} \tag{48} $$

where $\{\lambda^a\}$ denote the (Hermitian) generators of the group, $g$ the gauge coupling constant and $m$ the fermion mass. It is clear from the above expression that the sum of the three limits vanishes by virtue of the Jacobi identity obeyed by the group generators $\lambda^a$. We therefore conclude that the constant $\bar{c}_J$ in (38) is zero to this order (see [20] for a related result).

## 5. Conclusions

We considered the simultaneous use of the operator product expansion (OPE) and the Bjorken-Johnson-Low (BJL) limit techniques to study double commutators and thus look into possible violations of the Jacobi identity. The advantages of the method are its non-perturbative nature, the fact that all symmetries are manifest at each stage of the calculation, and its calculational ease. The disadvantages of the method are

[5]This regulator induces several new vertices in the theory, but this does not affect the present calculation.

that all results are determined up to unknown multiplicative constants which could, in fact, be zero (in which case no violations of the Jacobi identity appear). We note, however, that the vanishing of such constants would be accidental in the sense that it is not mandated by any symmetry of the model.

We were able to isolate cases where there cannot be violations of the Jacobi identity. For example the vacuum expectation value of $\langle 0| \mathcal{J}[J_0^a, J_0^b, J_0^c] |0\rangle = 0$ (when there is no symmetry breaking), as discussed in section 4.1. As another example one can consider a 4-dimensional gauge theory with a scalar field $\phi$; in this case $\mathcal{J}[\phi, \partial_\mu \phi, A_\nu] = 0$ identically.

As mentioned above the method proposed provides a *definition* of the Jacobi operator $\mathcal{J}[A, B, C]$ in (1). This definition, coincides with the naive expression (*i.e.*, it vanishes) whenever the operators $A$, $B$, $C$, and their triple products are well defined. When regularization is needed the expression for $\mathcal{J}$ need not vanish. The origin of this effect can be seen as follows: the naive expression for $\mathcal{J}$ contains two terms of the form $ABC$, one from the first double commutator in (1), and one from the last double commutator. The the first equal-time commutator, however, is evaluated by first letting $t_B \to t_A$ and subsequently $t_C \to t_A$; the second commutator is obtained by taking $t_C \to t_A$ and then $t_B \to t_A$. The two limits need not commute leading to a non-zero contribution. This can be interpreted as a lack of associativity, $(AB)C \neq A(BC)$ which is related to the presence of a three cocycle. A naive definition of $\mathcal{J}$ would not exhibit this feature, the cost being that the operator products are ill defined.

As applications we considered violations of the Jacobi identity for three chromo-electric fields as well as for three non-Abelian charges and for three Gauss' law generators. The resulting three-cocycles satisfy the closure relation (3) and therefore imply that the corresponding group is not associative. The general analysis relates the violations of the Jacobi identity to poles in the Wilson coefficient functions at large time-like momenta. Such poles are absent in most perturbative contributions leading to $\mathcal{J} = 0$; this is the case for three chromo-electric fields and three current charges. The one exception we have found corresponds to those perturbative contributions generated by the triangle graph, which generate non-trivial violations to the Jacobi identity of three (space-like) currents. The form of these perturbative results agree with that obtained using the OPE and BJL limit approach.

The fact that general considerations lead to a violation of the Jacobi identity implies, as mentioned previously, that a well-defined representation of such operators does not exist [11]. For example, a representation for the gauge field operators cannot be extended to include the $E_i^a$; these objects are then to be defined in terms of their commutators with space-time smeared operators.

It is also worth noticing that even if the Jacobi identity fails the corresponding group can still be made associative by an appropriate quantization of the 3-cocycle [10].

## Acknowledgments

We would like to thank R. Jackiw for various interesting comments. This work was supported in part through funds provided by the Department of Energy under contract DE-FG03-92ER40837.


# References


[1] J. Schwinger, Phys. Rev. Lett. **3**, 269 (1959). T. Goto and I. Imamura, Prog. Theor. Phys. **14**, 196 (1955).

[2] S.G. Jo, Phys. Lett. **163B**, 353 (1985); Nucl. Phys. **B259**, 616 (1985); Phys. Rev. **D35**, 3179 (1987). K. Fujikawa, Phys. Lett. **188B**, 115 (1987). S. Hosono and K. Seo, Phys. Rev. **D38**, 1296 (1988). G.W. Semenoff, *Phys. Rev. Lett.* **60**, 680 (1988); ERRATUM: *ibid.* **60**, 1590 (1988). T. Nishikawa and I. Tsutsui, Nucl. Phys. **B308**, 544 (1988). S. Ghosh and R. Banerjee, Phys. Lett. **B220**, 581 (1989). T.D. Kieu, Phys. Lett. **B223**, 72 (1989). W.T. Kim *et. al.*, Phys. Rev. **D42**, 2748 (1990). K. Seo, Phys. Rev. **D42**, 2819 (1990). G.V. Dunne, and C.A. Trugenberger, Phys. Lett. **B248**, 305 (1990).

[3] J. P. Muniain and J. Wudka, Phys. Rev. **D52**, 5194 (1995).

[4] K. G. Wilson, Phys. Rev. **179** 1499 (1969).

[5] J. D. Bjorken Phys. Rev. **148**, 1467 (1966). K. Johnson and F. E. Low Prog. Theor. Phys. **Suppl. 37-38**, 74 (1966).

[6] For a review see R. Jackiw in S. Treiman *et. al.*, *Current Algebra and Anomalies* (Princeton University Press, Princeton, N. J. , 1985.)

[7] R. Banerjee and H.J. Rothe, Int. J. Mod. Phys. **A6**, 5287 (1991)

[8] B. Zumino, in *Symp. on Anomalies, Geometry and Topology*, Argonne, IL, Mar 28-30, 1985. Edited by W.A. Bardeen and A.R. White (World Scientific, 1985). B. Zumino, Nucl. Phys. **B253**, 477 (1985). L.D. Faddeev, in *Recent developments in mathematical physics : proceedings of the XXVI Int. Universitatswochen fur Kernphysik*, Schladming, Austria, February 17-27, 1987. Edited by H. Mitter and L. Pittner. (Springer-Verlag, Berlin, New York, 1987). See also R. Jackiw, Ref. [6], G.W. Semenoff, Phys. Rev. Lett. **60**, 680 (1988).

[9] L.D. Faddeev, Phys. Lett. **145B**, 81 (1984). L.D. Faddeev and S.L. Shatashvili, Theor. Math .Phys. **60**, 770 (1985). J. Mickelsson, Phys. Rev. Lett. **54**, 2379

(1985). T. Fujiwara, Phys. Lett. **152B**, 103 (1985). L.D. Faddeev, Phys. Lett. **167B**, 225 (1986).

[10] R. Jackiw, Phys. Rev. Lett. **54**, 159 (1985). B. Grossman, Phys. Lett. **152B**, 93 (1985). Y.-S. Wu and A. Zee, Phys. Lett. **152B**, 98 (1985).

[11] A.L. Carey, Phys. Lett **B194**, 267 (1987). A.L. Carey and M.K. Murray, report HEPTH-9408141 (unpublished), e-Print Archive: hep-th/9408141 A.L. Carey *et.al.*, Comm. Math. Phys **168**, 389 (1995). A.L. Carey *et.al.*, J. Math. Phys. **36**, 2605 (1995).

[12] R. Jackiw, in *Quantum field theory and quantum statistics : essays in honor of the 60th birthday of E.S. Fradkin*, edited by I.A. Batalin *et.al.* (A. Hilger, Bristol, 1987). J. Wess, lectures given at *5th Adriatic Mtg. on Particle Physics*, Dubrovnik, Yugoslavia, Jun 16-28, 1986. Edited by M. Martinis and I. Andric. (World Scientific, Singapore, 1987). R. Floreanini and R. Percacci, Phys. Lett. **B231**, 119 (1989).

[13] F. Bucella et al, Phys. Rev. **149**, 1268 (1965).

[14] K. Johnson and F. E. Low Prog. Theor. Phys. **Suppl. 37-38**, 74 (1966).

[15] R.J. Crewther, Phys. Rev. Lett. **28**, 1421, (1972). S.L. Adler *et. al.*, Phys. Rev. **D6**, 2982 (1972).

[16] R.J. Eden *et.al.*, *The Analytic S Matrix*, (Cambridge, Cambridge Univ. Press, 1966.).

[17] A.D. Dolgov and V.I. Zakharov, Nucl. Phys. **B27**, 525 (1971). Y. Frishman *et.al.*, Nucl. Phys. **B177**, 157 (1981). J. Hořejší, Phys. Rev. **D32**, 1029 (1985).

[18] S.-G. Jo, Phys. Lett. **163B**, 353 (1985)

[19] For a recent review see T.D. Bakeev, A.A. Slavnov, report SMI-02-96; e-Print Archive: hep-th/9601092 (unpublished).

[20] T. Fujiwara *et.al.* Nucl.Phys. **B387**, 447 (1992). J. Kubo Nucl.Phys. **B427**, 398 (1994).